NST Part IA Mathematics

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1. Parallel and perpendicular components of a vector:
   $\mathbf{a}_{\parallel} = (\mathbf{a} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$
   $\mathbf{a}_{\perp} = \mathbf{a} - (\mathbf{a} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$

2. Vector triple product:
   $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
   $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}]$
   $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$

3. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

4. Plane equation:
   $\mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{a} \cdot \hat{\mathbf{n}} = d$
   $|d|$ is perpendicular distance of plane from origin for unit normal $\hat{\mathbf{n}}$

5. Polar coordinates:
   $\hat{\mathbf{r}} = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$
   $\hat{\boldsymbol{\phi}} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$
   $dS = rdrd\phi$

6. Cylindrical coordinates:
   $x = r\cos\phi$
   $y = r\sin\phi$
   $z = z$
   $dV = rdrd\phi dz$

7. Spherical coordinates:
   $x = r\sin\theta\cos\phi$
   $y = r\sin\theta\sin\phi$
   $z = r\cos\theta$

   $\hat{\mathbf{r}} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$
   $\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}$
   $\hat{\boldsymbol{\phi}} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$

   $dV = r^2\sin\theta drd\theta d\phi$
   $dS = r^2\sin\theta d\theta d\phi$

8. Leibnitz's formula:
   $$\frac{d^n(fg)}{dx^n} = \sum_{i=0}^{n}\binom{n}{i}f^{(n-i)}g^{(i)}$$

9. Limits:

   $\lim_{x\to a} f(x) = K$ means that $\forall\epsilon > 0.\exists\delta > 0.\ (0 < |x-a| < \delta) \implies (\ |f(x) - K| < \epsilon)$

   $\lim_{x\to a+} f(x) = K$ means that $\forall\epsilon > 0.\exists\delta > 0.\ (0 < x-a < \delta) \implies (\ |f(x) - K| < \epsilon)$

   $\lim_{x\to a-} f(x) = K$ means that $\forall\epsilon > 0.\exists\delta > 0.\ (0 < a-x < \delta) \implies (\ |f(x) - K| < \epsilon)$

   $\lim_{x\to\infty} f(x) = K$ means that $\forall\epsilon > 0.\exists X > 0.\ (x > X) \implies (\ |f(x) - K| < \epsilon)$

10. Continuity: $f(x)$ is continuous at a if:

    - $f(a)$ exists;
    - $\lim_{x\to a} f(x)$ exists and equals $f(a)$.

11. Taylor's series:

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots$$

12. Hyperbolic functions:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

13. Differentiation of integrals wrt parameters:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}dx + \frac{db}{dt}f(b,t) - \frac{da}{dt}f(a,t)$$

14. Schwarz's inequality:

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)$$

15. Gaussian integral:

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

16. Bayes' theorem:
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

17. Poisson distribution:
$$P(X=r) = e^{-\lambda}\frac{\lambda^r}{r!}$$
mean = variance = λ

18. Lifetime distribution:
$$f(t) = \lambda e^{-\lambda t}$$
mean = $\frac{1}{\lambda}$, variance = $\frac{1}{\lambda^2}$

19. Gaussian (normal) distribution:
$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
mean = μ, variance = σ^2

20. Linear 1st-order ODE:
$\frac{dy}{dx} + p(x)y = f(x)$:
$y = \frac{1}{\mu(x)} \int \mu(x) f(x) dx$, where $\mu(x) = e^{\int p(x)dx}$

21. Linear 2nd-order ODE (constant coefficients):
$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)$:
Complementary function: consider the auxiliary equation $\lambda^2 + a\lambda + b = 0$:
    - Case of real, distinct roots λ_1, λ_2: $y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
    - Case of equal roots α: $y_c = (A + Bx)e^{\alpha x}$
    - Case of complex roots $\alpha \pm \beta i$: $y_c = e^{\alpha x}(A \sin \beta x + B \cos \beta x)$

    Particular integral:
    - Case $f(x)$ is polynomial: try polynomial with same degree (or higher if needed)
    - Case $f(x) = Ce^{kx}$: try $y_p = De^{kx}$ (or Dxe^{kx}, or Dx^2e^{kx})
    - Case $f(x) = C_1 \sin kx + C_2 \cos kx$: try $y_p = D_1 \sin kx + D_2 \cos kx$ (or $D_1 x \sin kx + D_2 x \cos kx$)

22. Partial differentiation:
    - Differential relations:
    $df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$
    - Chain rule:
    $\left(\frac{\partial f}{\partial u}\right)_v = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$
    $\left(\frac{\partial f}{\partial v}\right)_u = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$
    - Reciprocity:
    $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1$
    - Cyclic relations:
    $\left(\frac{\partial x}{\partial z}\right)_y \left(\frac{\partial y}{\partial x}\right)_z \left(\frac{\partial z}{\partial y}\right)_x = -1$
    - Exact differential:
    $P(x,y)dx + Q(x,y)dy$ is an exact differential iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
    - Taylor series:
    $$\begin{aligned} f(x,y) &= f(x_0,y_0) \\ &\quad + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &\quad + \frac{1}{2!}\left(f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right) + \cdots \end{aligned}$$

23. Stationary points of multi-variable functions: f has a stationary point if $f_x = f_y = 0$
    - Local minimum: $f_{xx}f_{yy} > f_{xy}^2$ with $f_{xx} > 0$ and $f_{yy} > 0$
    - Local maximum: $f_{xx}f_{yy} > f_{xy}^2$ with $f_{xx} < 0$ and $f_{yy} < 0$
    - Saddle point: $f_{xx}f_{yy} < f_{xy}^2$

24. Lagrange multipliers and Lagrangian function:
To find the stationary points of $f(x,y)$ subject to the constraint $g(x,y)=0$:
define the Lagrangian function

$$L(x,y,\lambda)=f(x,y)-\lambda g(x,y)$$

and solve $L_x=L_y=L_\lambda=0$.

25. Gradient of a scalar field:
$\boldsymbol{\nabla}\Phi=(\Phi_x,\Phi_y,\Phi_z)$
$d\Phi=(\boldsymbol{\nabla}\Phi)\cdot d\mathbf{x}$

26. Divergence of a vector field:

$$\text{div }\mathbf{F}=\boldsymbol{\nabla}\cdot\mathbf{F}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(F_x,F_y,F_z)=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\Phi)=\nabla^2\Phi=\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}$$

27. Curl of a vector field:

$$\text{curl }\mathbf{F}=\boldsymbol{\nabla}\times\mathbf{F}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(F_x,F_y,F_z)=\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z},\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x},\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$$

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\Phi)=\mathbf{0}$$

28. Normal to a surface:

$$\mathbf{n}=\frac{\boldsymbol{\nabla}\Phi}{|\boldsymbol{\nabla}\Phi|}$$

29. Line integral of a scalar field:

$$\int_\Gamma \Phi ds=\int_{s_1}^{s_2}\Phi(\mathbf{x}(s))ds=\int_{t_1}^{t_2}\Phi(\mathbf{x}(t))\left|\frac{d\mathbf{x}}{dt}\right|dt$$

30. Line integral of a vector field:

$$\int_\Gamma \mathbf{F}(\mathbf{x})\cdot d\mathbf{x}=\int_{t_1}^{t_2}\mathbf{F}(\mathbf{x}(t))\cdot\frac{d\mathbf{x}}{dt}dt$$

31. Conservative vector fields: $\mathbf{F}=\boldsymbol{\nabla}\Phi$

$$\int_\Gamma \mathbf{F}\cdot d\mathbf{x}=\int_\Gamma(\boldsymbol{\nabla}\Phi)\cdot d\mathbf{x}=\Phi(\mathbf{x}_1)-\Phi(\mathbf{x}_2)$$

$$\oint_\Gamma \mathbf{F}\cdot d\mathbf{x}=\oint_\Gamma(\boldsymbol{\nabla}\Phi)\cdot d\mathbf{x}=0$$

32. Surface integral (flux):

$$\int_S \mathbf{F}\cdot d\mathbf{S}=\int_S \mathbf{F}\cdot\mathbf{n}dS$$

33. Gauss's theorem (divergence theorem):

$\displaystyle\int_V(\boldsymbol{\nabla}\cdot\mathbf{F})dV=\int_S\mathbf{F}\cdot d\mathbf{S}$, where S is the bounding surface of V

34. Stoke's theorem (curl theorem):

$\displaystyle\int_S(\boldsymbol{\nabla}\times\mathbf{F})\cdot d\mathbf{S}=\int_C\mathbf{F}\cdot d\mathbf{x}$, where C is the boundary of S (also called ∂S)

35. Decomposing matrix $\mathbf{M}$ as sum of a symmetric matrix $\mathbf{S}$ and an anti-symmetric matrix $\mathbf{A}$:
$\mathbf{S} = \frac{1}{2}(\mathbf{M} + \mathbf{M}^\top)$, $\mathbf{A} = \frac{1}{2}(\mathbf{M} - \mathbf{M}^\top)$
For an anti-symmetric matrix $\mathbf{A}$, $\mathbf{x}^\top \mathbf{A}\mathbf{x} = 0$ for any column vector $\mathbf{x}$.

36. Hermitian conjugation:
If $\mathbf{A} = (a_{ij})$, then the Hermitian conjugate is $\mathbf{A}^\dagger = (\mathbf{A}^\top)^* = (\mathbf{A}^*)^\top = (a^*_{ji})$
Hermitian matrix: $\mathbf{A}^\dagger = \mathbf{A}$

37. Trace: For an $n \times n$ matrix $\mathbf{A}$, trace($\mathbf{A}$)=$\displaystyle\sum_{i=1}^{n} a_{ii}$
The trace of the product of a symmetric and an antisymmetric matrix is 0.
trace($\mathbf{AB}$)=trace($\mathbf{BA}$).
The results can be generalised and holds for any cyclic permutation of the order of multiplication.

38. Minors and cofactors:
For an $n \times n$ matrix $\mathbf{A} = (a_{ij})$, let $\mathbf{M}_{ij}$ be an $(n-1) \times (n-1)$ sumathbfatrix:
   - Minor of the element a_{ij} of $\mathbf{A}$: $|\mathbf{M}_{ij}|$
   - Cofactor of a_{ij}: $A_{ij} = (-1)^{i+j}|\mathbf{M}_{ij}|$

$$\begin{pmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

   - Classical adjoint: (adj$\mathbf{A}$)$_{ij} = A_{ji}$

$$\begin{pmatrix} A_{11} & A_{21} & \dots & A_{j1} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{j2} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1i} & A_{2i} & \dots & A_{ji} & \dots & A_{ni} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{jn} & \dots & A_{nn} \end{pmatrix}$$

39. Determinants:

$|\mathbf{A}| = \displaystyle\sum_{j=1}^{n} a_{ij}A_{ij}$ for any fixed i; or
$|\mathbf{A}| = \displaystyle\sum_{i=1}^{n} a_{ij}A_{ij}$ for any fixed j; or

product of the elements on the diagonal if the matrix is triangular.
   - $\mathbf{A}$(adj$\mathbf{A}$) = (det$\mathbf{A}$)$\mathbf{I}$
   - Interchanging any two rows or columns of $\mathbf{A}$ changes the sign of det$\mathbf{A}$
   - det$\mathbf{A} = 0$ if any two rows or columns are the same
   - Multiplying all the elements of any one row or column of $\mathbf{A}$ by λ multiplies det$\mathbf{A}$ by λ
   - Adding a multiple of row (column) on another row (column) leaves det$\mathbf{A}$ unchanged
   - det$\mathbf{AB}$ = (det$\mathbf{A}$)(det$\mathbf{B}$)
   - det$\mathbf{A}$ = det$\mathbf{A}^\top$

40. Eigenvalues:
   - det$\mathbf{A} = \displaystyle\prod_{i=1}^{n} \lambda_i$
   - trace($\mathbf{A}$) $= \displaystyle\sum_{i=1}^{n} \lambda_i$

41. Diagonalisation of real symmetric matrices: if $\mathbf{A}$ is real symmetric, then:
    - $\mathbf{A}$ has n distinct eigenvalues λ_1, λ_2, ..., λ_n;
    - $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_n$ that form an orthonormal basis;
    - $\mathbf{A}$ can be diagonalised by setting $\mathbf{X} = \begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n\end{pmatrix}$, then

$$\mathbf{A}' = \mathbf{X}^\top \mathbf{A}\mathbf{X} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$